Summer Research Internship Project

Using Percolation to Analyze transport Properties of an Aquifier

Aritrabha Majumdar

Under the guidance of Professor Amit Ghoshal

Indian Institute of Science Education and Research, Kolkata

Contents

1 Introduction

Ground Water is stored in *aquifers*. There are usually two types of *aquifers* that can be seen, namely

1. **Saturated Aquifers:** In this case, all pore spaces or channels to transport water is *completely* filled with water.
2. **Unsaturated Aquifers:** In this case, pore spaces contain air and *mostly* they are partially filled with water.

In geological literatures, the hydrodynamical description (both microscopic and macroscopic) is available. Studies to understand the underlying flow structure is available for *saturated* and *homogenous* porous medium. [6] has detailed report in this regard for both *isotropic* and *anisotropic* porous media. We attempt to develop a *microscopic formulation* to model the aquifers and describe the transport properties for *inhomogeneous porous media aquifer.*
An *aquifer* is an underground layer of water-bearing material, such as permeable rock or unconsolidated materials like gravel, sand, or silt. It is capable of yielding significant quantities of water to wells and springs. But from a physical (even mathematical) point of view, an *aquifer* is a disorganized network of water channels. Therefore, our motivation is now to find one (or more!) suitable models to model the network. We need to model the aquifer with more 'extended' stringlike objects in stead of single bonds. Therefore, we resort to *random walks.* Eventually, we will consider a *specific* setting of random walk with some special property, and study the *extension* of walk and some other properties near *criticality.*

2 Some Probabilistic Backbone

2.1 Defining the Random Walk

We fix $d \in \mathbb{N}$ and take $||\cdot||$ to be the $\mathcal{L}^2$ norm on $\mathbb{R}^d$. Now, we observe the *random walk* on $\mathbb{Z}^d$ as (sum) of *random vectors* over a probability space $(\Omega_N, \mathbb{P})$, $N \in \mathbb{N}$. We write $\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) \mid \omega_k \in \mathbb{Z}^d, ||\omega_k|| = 1, \forall 1 \leq k \leq N\}$. X_k is defined as $X_k(\omega) = \omega_k$. The *displacement* is given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$, $1 \leq n \leq N$ and $S_0(\omega) = 0$ by definition. Observe that

$$S_n = \begin{pmatrix} S_n^{(1)} \\ S_n^{(2)} \\ \vdots \\ S_n^{(d)} \end{pmatrix}, \quad 0 \leq n \leq N, S_n^{(j)} \in \mathbb{Z}, j = 1, 2, \ldots, d$$

We use $\mathbb{E}(||S_n||^2)$ (mean squared displacement) as a measure of *extension.* (Consider [1] for more rigorous details.)

2.2 The Bond Percolation Model

For $d \in \mathbb{N}$, we can consider $\mathbb{Z}^d$ to be a subset of $\mathbb{R}^d$, and we take $\delta(\cdot)$ to be the $\mathcal{L}^1$ norm on $\mathbb{R}^d$. We may turn this setup to a graph by adding x and y whenever $\delta(x, y) = 1$. Thus we get the edge set E^d, and we define the graph to be $\mathbb{L}^d = (E^d, \mathbb{Z}^d)$. We fix $p \in [0, 1]$ and $e \in E^d$ is said to be *open* with probability p, and with probability $1 - p$. We define the sample space to be $\Omega = \prod_{e \in E^d}\{0, 1\}$, and $\omega(e) = 1$ if the edge is open, $\omega(e) = 0$ otherwise. If $\mathcal{F}$ is the σ algebra of subsets of Ω generated by finite dimensional cylinders, we take the product measure $\mathbb{P}_p = \prod_{e \in E^d} \mu_e$, where $\mu_e(\omega(e) = 1) = p$, $\mu_e(\omega(e) = 0) = 1 - p$. If $A \leftrightarrow B$ is a *path* from point A to point B, then one of the natural choice is to consider *self avoiding walks* on $\mathbb{Z}^2$. Now, we will rigorously define this and will analyze some of this properties near criticality ($p \in B_\varepsilon(p_c) \cap [0, 1]$), p_c is defined as $\mathbb{P}(\exists$ a path $0 \leftrightarrow \infty) = 1$, for $p > p_c$.

2.3 Defining the Self Avoiding Walk (SAW)

The rigorous definition of Self Avoiding Walk and some of its interesting quantitative properties can be found in [3] and in [2]. For each $n \in \mathbb{N}_0$, we define

$$\mathcal{W}_n = \{w = (w_i)_{i=0}^n \in \left(\mathbb{Z}^d\right)^{n+1} \mid w_0 = 0, ||w_{i+1} - w_i|| = 1 \forall 0 \leq i < n, w_i \neq w_j \forall i \neq j, \forall 0 \leq i < j \leq n\}$$

We also define the the walk measure on $\mathcal{W}_n$ as the *Gibbs Measure*, i.e

$$P_n(w) = \frac{1}{Z_n} e^{-\beta H_n(w)} \ \forall w \in \mathcal{W}_n$$

Here Z_n is the normalizing partition sum and H_n is the hamiltonian of the system. If we have $c_n = |\mathcal{W}_n|$ many random walks, then we uniformly choose one of them, i.e P_n assigns an *uniform* measure on $\mathcal{W}_n$. Therefore it is safe to assume $H_n \equiv 0$. For a physical description behind this assumption, we may assume that random walk moves with constant velocity, and it has no energy dependent restrictions, and a walk has no *interaction* with other walks.
The *connective constant* of $\mathbb{L}^d$ is given by $\mu(d) = \lim_{n \to \infty} (c_n)^{\frac{1}{n}}$, where c_n is number of self avoiding walks of length n.
The *free energy* of the self avoiding walk is given by $f = \lim_{n \to \infty} \frac{\log Z_n}{n}$. [2] shows that $f'(\beta)$ exists and has an exact expression.

Theorem: $\mu(d) \leq 2d - 1$

Proof: For SAW, the first step has altogether $2d$ options. The next steps has atmost $2d - 1$ many options, excluding its own position. Therefore $c_n \leq 2d(2d - 1)^{n-1}$. Now

$$\mu(d) = \lim_{n \to \infty} (c_n)^{\frac{1}{n}} \leq \lim_{n \to \infty} \left(\frac{2d}{2d - 1}\right)^{\frac{1}{n}} (2d - 1) = 2d - 1 \quad \blacksquare$$

3 Some Comments on the value of Critical Probability

It is known that $p_c(d) \sim \frac{1}{2d}$ as $d \to \infty$. This tells us that for large values of d, bond percolation on $\mathbb{L}^d$ behaves as bond percolation on a *regular tree*, where each vertex has $2d(1 + o(1))$ many descendants. It is also noteworthy that $\mathbb{L}^d$ can be embedded into $\mathbb{L}^{d+1}$ by means of *natural projection*. Therefore, for a fixed value of p, origin of $\mathbb{L}^{d+1}$ automatically belongs to an infinite cluster if origin of $\mathbb{L}^d$ belongs to an open cluster, therefore $p_c(d + 1) \leq p_c(d)$, $\forall d \geq 1$. It i obvious that for $d = 1$, $p_c = 1$, otherwise we have infinitely many closed edges on both side of the origin.
We define X_e to be a family of independent random variable indexed by $e \in E^d$. Now we make a *coupling* of all bonds available in $\mathbb{L}^d$. For $p \in (0, 1)$, we define $\psi_p \in \Omega$ as

$$\psi_p(e) = \begin{cases} 1 & \text{if } X_e < p \\ 0 & \text{if } X_e \geq p \end{cases}$$

We also define $\theta(p)$ as $\mathbb{P}$(any vertex (by translational invariance, the origin) is part of a infinite cluster), then for $p_1 \leq p_2$, $p_1 \leq p_2 \iff \psi(p_1) \leq \psi(p_2) \implies \theta(p_1) \leq \theta(p_2)$. So, $\theta(p)$ is a *non decreasing* function of p.

Theorem: For $d \geq 2$, $0 < p_c(d) < 1$.

Proof: Enough to show that $p_c(d) > 0\ \forall d \geq 2$ and $p_c(2) < 1$; as $p_c(d+1) \leq p_c(d)\ \forall d \geq 2$. Now; consider bond percolation on $\mathbb{L}^d$ for $d \geq 2$. Our aim is to show: $\theta(p) = 0$ when p is sufficiently close to 0 . Let $N(n)$ be the number of open paths starting at the origin. Any such path is open with probability p^n. Therefore

$$\mathbb{E}_p(N(n)) = p^n c_n$$

If origin is part of the infinite cluster; then there exists open path of every length starting at, the origin. Therefore

$$\theta(p) \leq \mathbb{P}_p(N(n) \geq 1) \leq \mathbb{E}_p(N(m)) = p^n c_n$$

Now, by definition $\mu(d) = \lim_{n\to\infty} \sigma(c_n)^{1/n} \implies c_n = (\mu(d) + \sigma(1))^n$. Now $n \to \infty$ gives us

$$\theta(p) \leq p^n(\mu(d) + o(1))^n = (p\mu(d) + o(1))^n$$

Now $\theta(p) \to 0$ when $n \to \infty$ if $p\mu(d) < 1 \implies p < \frac{1}{\mu(d)}$. Hence $p_c(d) \geq \frac{1}{\mu(d)} \implies p_c(d) > 0 \quad \forall d \geq 2$.

Now, to show $p_c(2) < 1$; we would use Peierls Argument. We shall show that $\theta(p) > 0$. if p is sufficiently close to 1. We construct the *planar dual* of $\mathbb{L}^2$ in usual manner; and let the vertex set be $\left\{x + \left(\frac{1}{2}, \frac{1}{2}\right); x \in \mathbb{Z}^2\right\}$; the edges are straight line segment in $\mathbb{R}^2$. An ledge of deed is closed if it cuts a closed edge; open of w. The dual is isomorphic to $\mathbb{L}^2$ and thus a bond percolation has been assigned on the dual. Suppose we have a finite open cluster containing the origin. The corresponding edges of the dual contain the origin in Interior of the closed circuit formed by the dual and vice versa.

Let $\rho(n)$ be number of circuits of length n containing origin of $\mathbb{L}^2$ in its interior. Each circuit passes through some vertex of the form $(k + 1/2, 1/2); 0 \leq k < n$. It happens as if surrounds the origin and $k < n$ as it has length n. Hence, the circuit contains a self avoiding walk of length $n - 1$ starting at $(k + 1/2, 1/2); 0 \leq k < n$, Hence

$$\rho(n) \leq nc_{n-1}$$

Let Γ be a closed circuit, $M(n)$ be number of closed circuits of length n. Now,

$$\begin{aligned}
\sum_{8} \mathbb{P}_p(\Gamma \text{ is closed}) &\leq \sum_{n=1}^{\infty} (1-p)^n nc_{n-1} \\
&= \sum_{n=1}^{\infty} (1-p)^n n((1-p)\mu(2) + o(1))^{n-1} \\
&< \infty \text{ if } (1-p)\mu(2) < 1
\end{aligned}$$

As $p \uparrow 1; \sum_\Gamma \mathbb{P}_p(\Gamma \text{ closed}) \to 0$. Hence we may find π satisfying $\pi \in (0,1)$ such that $\sum_\Gamma \mathbb{P}_p(\Gamma \text{ closed}) \le \frac{1}{2}$ when $p > \pi$.Therefore

$$\begin{aligned}\mathbb{P}_p(|C| = \infty) = \mathbb{P}_p(M(n) = 0 \forall n) &= 1 - \mathbb{P}_p(M(n) \ge 1 \text{ for some } n) \\ &\ge 1 - \sum_\Gamma \mathbb{P}_p(\Gamma \text{ is closed }) \\ &\ge \frac{1}{2} \quad \text{when } p > \pi\end{aligned}$$

$$\therefore p_c(2) \le \pi < 1 \quad \blacksquare$$

This theorem has given us an interesting observation. For $d = 2$, $p_c \ge \frac{1}{2}$. Later Kesten showed that $p_c \le \frac{1}{2}$ and this concluded $p_c = \frac{1}{2}$ for $d = 2$.

Theorem: $p_c(2) \leqslant 1 - \frac{1}{\lambda(2)}$

Proof: Define F_m to be the event that it contains a closed dual circuit with $B(m)$ is its interior; and G_m to be the event that all edges of $B(m)$ are open. Clearly F_m and G_m are defined on different edge space, $\therefore F_m \coprod G_m$. Now,

$$\mathbb{P}_p(F_m) \le \mathbb{P}_p\left(\sum_{n=4m}^{\infty} M(n) \ge 1\right) \le \sum (1-P)^n n c_{n-1} = \sum n(1-p)(1-p)\lambda(2) + o(1n)$$

As $p \uparrow 1; \mathbb{P}_p(F_m) \to 0$; By similar logic as before; $\mathbb{P}_p(F_m) < 1/2$ whenever $(1-p) < \frac{1}{\lambda(2)} \Rightarrow p > 1 - \frac{1}{\lambda(2)}$

Now: we are interested for the bound of $P_c(2)$. Consider $G_m \cap F_m^c$, which implies all edges of $B(m)$ are open and it's not in the interior of closed dual circuit. Therefore some vertex of $B(m)$ is member of an infinite open path $\Rightarrow |C| = \infty$ Hence

$$\theta(p) \ge \mathbb{P}_p(F_m^c \cap G_m) = \mathbb{P}_p(F_m^c)\,\mathbb{P}_p(G_m) \ge \frac{1}{2}\mathbb{P}_p(G_m) > 0$$

whenever $p > 1 - \frac{1}{\lambda(2)}$. Therefore $p_c(2) \leqslant 1 - \frac{1}{\lambda(2)}$ $\blacksquare$

Theorem: If $\Psi(p)$ denotes the probability that there exists an infinite open cluster, it satisfies

$$\Psi(p) = \begin{cases} 0 & \theta(p) = 0 \\ 1 & \theta(p) > 0 \end{cases}$$

Proof: If $\theta(p) = 0$, then

$$\Psi(p) \le \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(|C| = \infty) = 0 \implies \Psi(p) = 0$$

On the other hand, if $\theta(p) > 0$, then

$$\Psi(p) \ge \mathbb{P}_p(|C| = \infty) > 0$$

, Then by Borel-Cantelli Lemma, $\Psi(p) = 1$.

4 Some Analytical Result of Some Physical Properties

4.1 Free Energy

We have already defined the *free energy* to be

$$f = \lim_{n\to\infty} \frac{\log Z_n}{n}$$

First of all, we will show that that the limit exists. It is well known that $Z_{m+n} \le Z_m Z_n \implies \log Z_{m+n} \le \log Z_m + \log Z_n$. As it is a subadditive sequence, by Fekete's lemma,

$$f = \lim_{n\to\infty} \frac{\log Z_n}{n} = \inf_{n\in\mathbb{N}} \frac{\log Z_n}{n} \text{ exists in } [-\infty, \infty)$$

Now, we assume that the hamiltonian depends on a single parameter $\beta \in \mathbb{R}$ linearly. Therefore by applying Hölder's inequality, we get

$$Z_n\left(\frac{\beta_1}{2} + \frac{\beta_2}{2}\right) = \mathbb{E}_n\left(e^{\frac{\beta_1}{2}+\frac{\beta_2}{2}H_n}\right) \le \sqrt{\mathbb{E}_n(e^{\beta_1 H_n})}\sqrt{\mathbb{E}_n(e^{\beta_2 H_n})} = \sqrt{Z_n(\beta_1)}\sqrt{Z_n(\beta_2)}$$

If we set $f_n = \frac{\log Z_n}{n}$, this leaves us with

$$f_n\left(\frac{\beta_1}{2} + \frac{\beta_2}{2}\right) \le \frac{1}{2} f_n(\beta_1) + \frac{1}{2} f_n(\beta_2)$$

So, $f_n(\beta)$ is indeed convex, and taking the limit $n \to \infty$, we are left with $f(\beta)$ is convex. Convexity implies continuity, and at those values of β for which f is not differentiable, it indicates a *phase transition*. But if it is differentiable, then

$$f'(\beta) = \lim_{n\to\infty} f_n'(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n'(\beta) = -\lim_{n\to\infty} \frac{1}{n} \sum_{w\in\mathcal{W}_n} H_n(w) P_n(w)$$

4.2 Number of Walks

For SAW, we have seen that $H_n \equiv 0$, and that leaves us with $f'(\beta) = 0 \implies f(\beta)$ is constant. Now, we have $\log\mu = \lim_{n\to\infty} \frac{1}{n}\log c_n$, and again by Fekete's lemma, this limit exists and is equal to $f(\beta)$, as $c_n = Z_n$ for SAW. The Hammersely-Welsh bound for c_n serves as one of the most fundamental results. The bound says that $c_n \le \mu^n \exp[O(\sqrt{n})]$. The paper by [5] gives a simpler proof with a suboptimal constant in the exponent, and refines the bound by $c_n \le \exp\left[O(n^{(\nu-1)/(2\nu-1)})\right]\mu^n$

Fix $d \ge 2$. Our starting point will be the following inequality between generating functions. We define

$$\chi(z) = \sum_{n\ge 0} z^n c_n \quad \text{and} \quad B(z) = \sum_{n\ge 0} z^n b_n$$

to be the generating functions of self-avoiding walks and self-avoiding bridges respectively. We define $z_c = \mu_c^{-1}$, which is the radius of convergence of $\chi(z)$ and hence also of $B(z)$ by the following proposition.

Proposition: $\chi(z) \le z^{-1}\exp[2B(z) - 2]$ for every $z \ge 0$.
This inequality relies on similar ideas as the proof of the Hammersley-Welsh bound, but is easier to prove. It does not rely on the combinatorial analysis of the 'unfolding' of walks. We will prove that the following

version of Hammersley-Welsh with a suboptimal constant can be deduced directly from Proposition 2.1 by elementary methods.
Proposition. $c_n \leq \exp[\sqrt{8n} + O(\sqrt{1/n})]\mu_c^{n+1}$.
Let $L_n = \mathbb{Z}^{d-1} \times \{n\}$ for each $n \geq 0$. For each $z \geq 0$ and $n \geq 0$, define

$$a(z;n) = \sum_{\omega \in \Omega} z^{|\omega|} \mathbb{1}[\omega : 0 \to L_n \text{ is a SAB}]$$

If $\omega_1 : 0 \to L_n$ and $\omega_2 : 0 \to L_m$ are bridges, then we can form a bridge $\omega : 0 \to L_{n+m}$ by applying a translation to ω_2 so that it starts at the endpoint of ω_1 and then concatenating the two paths. This implies that the sequence $a(z;n)$ is supermultiplicative for each $z \geq 0$, meaning that

$$a(z;n+m) \geq a(z;n)a(z;m)$$

for every $n, m \geq 0$. It follows by Fekete's lemma that for each $z \geq 0$ there exists $\xi(z)$ such that

$$\xi(z) = \lim_{n\to\infty} -\frac{1}{n} \log a(z;n) \in [-\infty, \infty]$$

and that

$$a(z;n) \leq e^{-\xi(z)n}$$

for every $n \geq 0$.
Lemma:. $\xi\left(z_c\right) \geq 0$.
Proof: For each $n \geq 0, a(z;n)$ is expressible as a power series in z with non-negative coefficients, and is therefore left-continuous in z for $z > 0$. If $z < z_c$ then $B(z) = \sum_{n\geq 0} a(z;n) < \infty$, so that $\xi(z) \geq 0$ by the identity (2.1), and hence that $a(z;n) \leq 1$ for every $n \geq 0$ by the inequality (2.2). Since this bound holds for all $z < z_c$, it also holds for $z = z_c$ by left continuity.

A similar analysis shows that $(b_n)_{n\geq 0}$ is supermultiplicative and that

$$b_n \leq \mu_c^n$$

for every $n \geq 0$
We have the trivial inequality

$$a\left((1-\varepsilon)z_c;n\right) \leq (1-\varepsilon)^n a\left(z_c;n\right) \leq (1-\varepsilon)^n$$

It follows that $B\left((1-\varepsilon)z_c\right) \leq \varepsilon^{-1}$, and hence that

$$\chi\left((1-\varepsilon)z_c\right) \leq \frac{1}{z} \exp\left[2\varepsilon^{-1} - 2\right]$$

To conclude, we apply the trivial inequality

$$z_c^n c_n \le (1-\varepsilon)^{-n} \chi\left((1-\varepsilon)z_c\right)$$

with $\varepsilon = (n/2)^{-1/2}$ to deduce that

$$z_c^{n+1} c_n \le \exp[\sqrt{2n} - 2](1 - \sqrt{2/n})^{-n-1} = \exp[\sqrt{8n} + O(\sqrt{1/n})]$$

as $n \to \infty$, where the equality on the right-hand side follows by calculus (the -2 has not been forgotten).

Lemma: $\xi\left((1-\varepsilon)z_c\right) \ge -\psi(\varepsilon)\log(1-\varepsilon)$ for every $\varepsilon > 0$.
Proof: It suffices to prove that

$$\xi\left((1-\varepsilon)z_c\right) \ge \min\left\{-\lambda\log(1-\varepsilon), -\log(1-\varepsilon) + \Phi\left(\lambda^{-1}\right)\right\}$$

for every $\varepsilon > 0$ and $\lambda \ge 1$, since the result then follows by optimizing over λ. Splitting the walks contributing to $a\left((1-\varepsilon)z_c; n\right)$ according to whether they have length more than λn or not yields that

$$a\left((1-\varepsilon)z_c; n\right) \le \sum_{m \ge \lambda n} \sum_{\omega \in \Omega} z_c^m (1-\varepsilon)^m \mathbb{1}\left[\omega : 0 \to L_n \text{ a length } m\text{SAB}\right] + \sum_{m=n}^{\lambda n} \sum_{\omega \in \Omega} z_c^m (1-\varepsilon)^m \mathbb{1}\left[\omega : 0 \to L_n \text{ a length } m\text{SAB}\right]$$

and hence that

$$\begin{aligned} a\left((1-\varepsilon)z_c; n\right) &\le (1-\varepsilon)^{\lambda n} \sum_{m \ge \lambda n} \sum_{\omega \in \Omega} z_c^m \mathbb{1}\left[\omega : 0 \to L_n \text{ a length } m\text{SAB}\right] \\ &+ (1-\varepsilon)^n \sum_{m=n}^{\lambda n} \sum_{\omega \in \Omega} z_c^m \mathbb{1}\left[\omega : 0 \to L_n \text{ a length } m\text{SAB}\right] \end{aligned}$$

This implies that

$$\begin{aligned} a\left((1-\varepsilon)z_c; n\right) &\le (1-\varepsilon)^{\lambda n} a(z; n) + (1-\varepsilon)^n \sum_{m=n}^{\lambda n} z_c^m b_m \exp\left[-\phi\left(\frac{n}{m}\right) m\right] \\ &\le (1-\varepsilon)^{\lambda n} + \lambda n (1-\varepsilon)^n \exp\left[-\Phi\left(\lambda^{-1}\right) n\right] \end{aligned}$$

for every $n \ge 0$, applying the previous results yields the estimate

$$B\left((1-\varepsilon)z_c\right) = \sum_{n \ge 0} a\left((1-\varepsilon)z_c; n\right) \le \sum_{n \ge 0} e^{\psi(\varepsilon)\log(1-\varepsilon)n} = \left[1 - (1-\varepsilon)^{\psi(\varepsilon)}\right]^{-1}$$

We deduce from Proposition that

$$\chi\left((1-\varepsilon)z_c\right) \le \frac{1}{(1-\varepsilon)z_c} \exp\left[2\left[1 - (1-\varepsilon)^{\psi(\varepsilon)}\right]^{-1} - 2\right]$$

The claim now follows by applying the inequality as in the proof of aforementioned proposition
Proof of Theorem 1.2. Let ϕ be as in Theorem 1.3. It suffices to prove that $\Psi(n)=o\left(n^{1/2}\right)$. Since $\phi(\varepsilon)$ is increasing and $\phi(\varepsilon)>0$ for all $0<\varepsilon\leq 1$, we have that $\Phi(\varepsilon)>0$ for all $0<\varepsilon\leq 1$, and hence that $\psi(\varepsilon)\to\infty$ as $\varepsilon\to 0$. Fix $M\geq 1$ and $\delta>0$. Then there exists $0<\varepsilon_0\leq 1$ such that $\psi(\varepsilon)\geq M$ for every $0<\varepsilon\leq\varepsilon_0$. Setting $\varepsilon=\delta n^{-1/2}$ yields that

$$\Psi(n)\leq 2\left[1-\left(1-\delta n^{-1/2}\right)^{M}\right]^{-1}-(n+1)\log\left(1-\delta n^{-1/2}\right)$$

for every sufficiently large n. It follows that

$$\Psi(n)\leq\left(\delta+\frac{2}{\delta M}\right)n^{1/2}+o\left(n^{1/2}\right)$$

as $n\to\infty$. The result follows since $M\geq 1$ and $\delta>0$ were arbitrary (e.g. by taking $M=\delta^{-2}$ and sending $\delta\to 0$). Suppose that $\phi(\varepsilon)\geq C\varepsilon^{\nu}$ for some $C>0$ and $\nu>1$. Then $\Phi(\varepsilon)\geq C\varepsilon^{\nu-1}$ and hence

$$\psi(\varepsilon)\geq\sup\left\{\lambda\geq 1:\varepsilon\leq 1-\exp\left[-C\frac{\lambda^{1-\nu}}{\lambda-1}\right]\right\}$$

A straightforward analysis then yields that

$$\psi(\varepsilon)\geq C'\varepsilon^{-1/\nu}$$

for some $C'>0$ and every $\varepsilon>0$ sufficiently small. Let $\alpha>0$. Then

$$\Psi(n)\leq 2\left[1-\left(1-n^{-\alpha}\right)^{C'n^{\alpha/\nu}}\right]^{-1}-(n+1)\log\left(1-n^{-\alpha}\right)$$

for every n sufficiently large. It follows by calculus that

$$\Psi(n)=O\left(n^{\alpha(\nu-1)/\nu}+n^{1-\alpha}\right)$$

as $n\to\infty$. Optimizing by taking $\alpha=\nu/(2\nu-1)$ yields the improvement.

4.3 Mean Square Displacement

The Mean Square Displacement for any walk is defined by

$$\mathbb{E}(||S_n^2||)=\frac{1}{c_n}\sum_{w\in\mathcal{W}}S_n^2$$

At first, it was conjectured that

$$\mathbb{E}(||S_n||^2)=\begin{cases}Dn^{2\upsilon} & d\neq 4\\ Dn(\log n)^{\frac{1}{4}} & d=4\end{cases}$$

For two dimensions, the value of υ is 0.75. The conjectures were made long before, but a recent proof was given by [4]. Suppose that a self-avoiding walk (SAW) of length n has radius L. Then its density is $\approx n/L^d$ monomers per site (where $\approx$ means that the ratio of the two sides is bounded above and below by strictly positive and finite constants). Therefore, its "repulsive energy" per site is $\approx (n/L^d)^2$, and so its total repulsive energy is $\approx n^2/L^d$. Now, the probability that a simple random walk (SRW) has radius L is roughly $\exp(-L^2/n)$. Hence, the probability that a SAW has radius L is roughly

$$\exp\left(-\left\{\frac{n^2}{L^d}+\frac{L^2}{n}\right\}\right).$$

Put $L = n^\upsilon$. Then the term between braces equals

$$n^{2-d\upsilon}+n^{2\upsilon-1},$$

which is minimized when $2 - d\upsilon = 2\upsilon - 1$, or

$$\upsilon = \frac{3}{d+2}.$$

For this choice of υ, the term between braces takes the value $n^{(4-d)/(d+2)}$, and so we need $1 \leq d \leq 4$ for the total repulsive energy not to vanish. For $d \geq 5$, the total repulsive energy vanishes and we are in the diffusive regime with $\upsilon = \frac{1}{2}$.

4.4 Mean Cluster Size

The mean cluster size of an open cluster containing the origin is given by

$$\chi(p) = \mathbb{E}_p(C_0) = \sum_{r=1}^{\infty} \mathbb{P}_p(0 \leftrightarrow \partial B(r))$$

. But this definition needs a bit more clarification. By $\partial B(r)$, we usually represent the boundary of $[-r, r]^d$, a d dimensional box in $\mathbb{Z}^d$ of side length $2r$ around the origin. If a self avoiding path starting from the origin touches this *box* $B(r)$ for all $r \in \mathbb{N}$, it would give us an open cluster. Now, we would like to prove that $\chi(p) \sim |p - p_c|^{-\gamma}$.

Observation in one dimension: We define $\Lambda(p)$ as number of r cluster per lattice site when each bond is open (or each site is occupied) with probability p. Now for an r cluster to be present, we want r consecutive bonds to be open, and the two bonds at the two ends to be closed. So, $\Lambda(p) = p^r(1-p)^2$. The probability that an arbitrary site belongs to a r cluster is simply $r\Lambda(p)$, and the probability that a site is occupied is p. So the probability that the cluster to which an occupied sites belongs contains r sites is $\frac{r\Lambda(p)}{p}$. Now, the mean cluster size is

$$\chi(p) = \sum_{r=1}^{\infty} \frac{r^2\Lambda(p)}{p} = \frac{(1-p)^2}{p}\sum_{r=1}^{\infty} r^2 p^r = \frac{1+p}{1-p}$$

In one dimension, $p_c = 1$, therefore $\chi(p) \sim |p - p_c|^{-1}$. And, it is expected to follow similar kind of power law behavior in higher dimensions as well.

Using power law ansatz of correlation length: The definition of correlation length will be given in the next section in greater detail. But if we assume that correlation length also exhibits *power law behavior*, i.e. $\zeta(p) \sim |p - p_c|^{-\nu}$, (note that this ν is different from what we had in section 4.2), then we can show $\gamma = \nu(2 - \eta)$.

A *two point correlation function* $G(r)$ is defined as the extent of linear relationship at points present in distance r. In the setting of bond percolation model, this quantity is actually equal to $\mathbb{P}_p(0 \leftrightarrow \partial B(r))$. Now, we know that $G(r) \sim \frac{e^{-\frac{r}{\zeta}}}{r^{\text{exponent}}}$, we set this *exponent* to be $d - 2 + \eta$ for bond percolation in $\mathbb{Z}^d$. Therefore it can be written that

$$\chi(p) \sim \int_{-\infty}^{\infty} G(r)\, dx_1\, dx_2\, \ldots dx_d = \int_0^{\infty} \int_{\Theta} G(r) r^{d-1} f(\theta)\, dr\, d\theta$$

Here, $r^{d-1} f(\theta)$ is the jacobian, and the integral of the azimuthal part is independent of the linear part. At $p = p_c$, $\zeta \to \inf$. Therefore, applying Fubini's theorem, we can write

$$\chi(p) \sim \int_0^{\infty} r^{1-\eta} e^{r\zeta}\, dr = \int_0^{\zeta} (r)^{1-\eta}\ dr = \frac{\zeta^{2-\eta}}{2-\eta} \sim \zeta^{2-\eta}$$

So, we now have

$$|p - p_c|^{-\gamma} = (|p - p_c|^{-\nu})^{2-\eta} \implies \gamma = \nu(2 - \eta)$$

.

4.5 Correlation Length

Intuitively, the correlation length can be defined as $\zeta(p) = \sup\{R \mid \mathbb{P}(0 \leftrightarrow \partial B(R)) \geq \varepsilon\ \forall \varepsilon > 0\}$. But [3] gives us a more *mathematically accurate* description of correlation length.

Theorem: we have two independent constant ρ and σ independent of p and a function $\varphi(p)$ such that

$$\rho r^{1-d} e^{-n\varphi(p)} \leq \mathbb{P}_p(0 \leftrightarrow \partial B(r)) \leq \sigma r^{d-1} e^{-n\varphi(p)}$$

Proof: Let $0 < p \leq 1$, and let $\beta(n) = \mathbb{P}_p(0 \leftrightarrow \partial B(n))$ and $\tau_p(0, x) = \mathbb{P}_p(0 \leftrightarrow x)$. Now, using BK inequality, we can write

$$\mathbb{P}_p(0 \leftrightarrow \partial B(m+k)) \leq \sum_{x \in \partial B(m)} \mathbb{P}_p(0 \leftrightarrow x) \mathbb{P}_p(x \leftrightarrow \partial B(k, x)) = \mathbb{P}_p(0 \leftrightarrow \partial b(k)) \sum_{x \in \partial B(m)} \tau_p(0, x)$$

Observe that when $x \in \partial B(m)$, $\tau_p(0, x) \leq \beta(m)$. Therefore we rewrite the previously obtained relation as

$$\beta(m+n) \leq |\partial B(m)| \beta(m) \beta(n)$$

Let $\gamma(n)$ be the probability that there exists some vertex $x \in \partial B(n)$ with $x_1 = n$ which is joined to the origin by an open path of $B(n)$. We have defined $\gamma(n)$ in terms of a particular face of $B(n)$, but clearly this probability is the same for each of the $2d$ faces of $B(n)$. Thus

$$\gamma(n) \leq \beta(n) \leq 2d\gamma(n).$$

Let $x \in \partial B(m)$, and choose k such that $x_k = \pm m$; we shall suppose that $x_k = m$, and an analogous argument is valid if $x_k = -m$. Let U_x be the event that there exists an open path in $B(m)$ joining x to the origin, and let V_x be the event that there exists an open path in $B(n, x)$ joining x to some vertex y of $\partial B(n, x)$ for which $y_k = m + n$; see Figure 6.3. Now,

$$\beta(m+n) \geq \mathbb{P}_p(U_x \cap V_x),$$

since the union of two such paths contains a connection from the origin to the surface $\partial B(m+n)$. We use the FKG inequality to find that

$$\beta(m+n) \geq \mathbb{P}_p(U_x) \mathbb{P}_p(V_x).$$

Now, $\mathbb{P}_p(V_x) = \gamma(n)$, and

$$\beta(m) = \mathbb{P}_p\left(\bigcup_{x\in\partial B(m)} U_x\right) \leq \sum_{x\in\partial B(m)} \mathbb{P}_p(U_x),$$

giving that there exists $x \in \partial B(m)$ such that

$$\mathbb{P}_p(U_x) \geq \frac{1}{|\partial B(m)|}\beta(m).$$

With this choice of x, it becomes

$$\beta(m+n) \geq \frac{1}{|\partial B(m)|}\beta(m)\gamma(n) \geq \frac{1}{2d|\partial B(m)|}\beta(m)\beta(n)$$

Now, for the finer details present, we have

$$|\partial B(m)| \leq 2d\left|\left\{x \in \mathbb{Z}^d : x_1 = m,\ |x_i| \leq m \text{ for } 2 \leq i \leq d\right\}\right| = 2d(2m+1)^{d-1} \leq d^3 m^{d-1} \quad \text{for } m \geq 1,$$

and then we rewrite the previous inequalities in the form

$$\log\beta(m+n) \leq \log\beta(m) + \log\beta(n) + g(m),$$

$$\log\beta(m+n) \geq \log\beta(m) + \log\beta(n) - g(m),$$

where

$$g(r) = \log\left(d^2 3^{d+1} r^{d-1}\right) = \log\left(d^2 3^{d+1}\right) + (d-1)\log r.$$

There are at least two ways of applying the subadditive limit theorem to these inequalities, and one such possibility is to note from the monotonicity of g that the sequences $(\log\beta(n) : n \geq 1)$ and $(-\log\beta(n) : n \geq 1)$ satisfy the generalized subadditive inequality with error function g. An alternative argument obviates the need to appeal to this general version of the subadditive limit theorem. Suppose that $m \leq n$, and add $g(n)$ to both sides of the previously obtained result to get

$$g(n) + \log\beta(m+n) \leq \{g(m) + \log\beta(m)\} + \{g(n) + \log\beta(n)\}.$$

Now

$$g(m+n) - g(n) = (d-1)\log\left(1 + \frac{m}{n}\right) \leq (d-1)\log 2$$

since $m \leq n$. Adding these two inequalities, we find that the sequence $(a_k : k \geq 1)$, defined by

$$a_k = g(k) + (d-1)\log 2 + \log\beta(k),$$

satisfies the subadditive inequality

$$a_{m+n} \leq a_m + a_n \quad \text{for } m, n \geq 1.$$

Therefore again by Fekete's Lemma, we know that

$$\varphi(p) = \lim_{k\to\infty}\left\{\frac{\log\beta(k)}{k}\right\}$$

exists, and we have $|m\varphi(p) + \log\beta(m)| \leq g(m) + (d-1)\log 2$, proving our argument.

Theorem: $\varphi(p) > 0$ when $p \in (0, p_c)$, 0 otherwise. Moreover it is strictly decreasing on $(0, p_c)$. $\varphi(p) \to \infty$ as $p \downarrow 0$.

Proof: Here, we use a theorem: let A be an increasing event depending only on the finitely many edges of $\mathbb{L}^d$ and $p \in (0,1)$. Then $\frac{\log \mathbb{P}_p(A)}{\log p}$ is a non-increasing function of p.

First we prove that φ is continuous on $(0,1)$. Let

$$b_m(p) = -\frac{1}{m} \log \beta(m),$$

and write in the form

$$|\varphi(p) - b_m(p)| \leq \frac{1}{m} \left[c + (d-1) \log m\right].$$

Now $b_m(p) \to \varphi(p)$ as $m \to \infty$, and this convergence is uniform on $(0,1]$ since the right side does not involve p. Also, $b_m(p)$ is a continuous function of p, since the event $\{0 \leftrightarrow \partial B(m)\}$ depends only on the finite collection of edges within $B(m)$. Thus $\varphi(p)$ is the uniform limit of continuous functions, and therefore φ is continuous also. We have proved also that $\varphi(p_c) = 0$, since

$$\varphi(p_c) = \lim_{p \uparrow p_c} \varphi(p) \quad \text{(by continuity)} = 0.$$

The strict monotonicity of φ on $(0, p_c]$ is proved by applying to the event $\{0 \leftrightarrow \partial B(n)\}$. This is an increasing event which depends only on the edges in $B(n)$, and it is a consequence of the mentioned theorem that

$$\frac{\log P_a(0 \leftrightarrow \partial B(n))}{\log a} \geq \frac{\log P_b(0 \leftrightarrow \partial B(n))}{\log b} \quad \text{if } a \leq b.$$

We divide by n and take the limit as $n \to \infty$ to deduce that

$$\frac{\varphi(a)}{\log a} \leq \frac{\varphi(b)}{\log b} \quad \text{if } 0 < a \leq b \leq 1;$$

hence

$$\varphi(a) \geq \varphi(b) \frac{\log(1/a)}{\log(1/b)} \quad \text{if } 0 < a \leq b \leq 1.$$

Finally, we show that $\varphi(p) \to \infty$ as $p \downarrow 0$, and we use the path-counting argument:

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n)) \leq \mathbb{P}_p(N(n) \geq 1) \leq \{p\lambda(d) + o(1)\}^n,$$

where $N(n)$ is the number of open paths starting at the origin and $\lambda(d)$ is the connective constant of $\mathbb{L}^d$. This gives us

$$\varphi(p) \geq -\log\{p\lambda(d)\} \to \infty \quad \text{as } p \downarrow 0.$$

Now we observe some interesting results. Using the notation $e_n = (n, 0, 0, \ldots, 0)$, we can write

$$\{0 \leftrightarrow e_{m+n}\} \supseteq \{0 \leftrightarrow e_m\} \cap \{e_m \leftrightarrow e_{m+n}\};$$

By the FKG inequality and translation invariance,

$$\tau_p(0, e_{m+n}) \geq \tau_p(0, e_m)\tau_p(e_m, e_{m+n}) = \tau_p(0, e_m)\tau_p(0, e_n).$$

Thus $t(k) = -\log \tau_p(0, e_k)$ satisfies
$$t(m+n) \leq t(m) + t(n),$$
and the Fekete's lemma gives us that the limit
$$\eta(p) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \tau_p(0, e_n) \right\}$$
exists. Furthermore,
$$\log \tau_p(0, e_n) = t(n) \leq -n\eta(p) \quad \text{for all } n.$$

We show next that $\eta(p) = \varphi(p)$. First note that
$$\tau_p(0, e_n) \leq \mathbb{P}_p(0 \leftrightarrow \partial B(n)),$$
This implies that $\eta(p) \geq \varphi(p)$. It requires slightly more work to show the opposite inequality. We have that there exists a vertex $x \in \partial B(m)$ such that
$$\mathbb{P}_p(0 \leftrightarrow x \text{ in } B(m)) \geq \frac{1}{|\partial B(m)|} \mathbb{P}_p(0 \leftrightarrow \partial B(m)).$$

Using the rotation invariance of the lattice, we may pick such a vertex x satisfying $x_1 = m$. By symmetry, the events $\{0 \leftrightarrow x\}$ and $\{x \leftrightarrow e_{2m}\}$ are equally likely (just reflect the set of paths from 0 to x in the hyperplane $\{y \in \mathbb{Z}^d : y_1 = m\}$ to obtain the isomorphic set of paths from e_{2m} to x). On the other hand,
$$\{0 \leftrightarrow e_{2m}\} \supseteq \{0 \leftrightarrow x\} \cap \{x \leftrightarrow e_{2m}\}$$

We use the FKG inequality here to obtain
$$\tau_p(0, e_{2m}) \geq \tau_p(0, x)^2,$$

which we combine with the previous estimate to obtain
$$\tau_p(0, e_{2m}) \geq A_1 m^{2(1-d)} \mathbb{P}_p(0 \leftrightarrow \partial B(m))^2 \geq A_2 m^{4(1-d)} e^{-2m\varphi(p)} \quad \text{by (6.11)},$$

for appropriate positive constants A_1 and A_2; we have used the fact that $|\partial B(m)|$ is of order m^{d-1}. This leaves us with $\eta(p) = \varphi(p)$ If $n = 2m+1$, then, with the same choice of x,
$$\{0 \leftrightarrow e_{2m+1}\} \supseteq \{0 \leftrightarrow x\} \cap \{x, x+e_1 \text{ is open}\} \cap \{x + e_1 \leftrightarrow e_{2m+1}\},$$
an intersection of increasing events. Applying the FKG inequality, we obtain that
$$\tau_p(0, e_{2m+1}) \geq p\tau_p(0, x)^2,$$
and the result follows as before. So, we are left with another useful representation of $\varphi(p)$. All these result motivates us to define correlation length as $\zeta(p) = \varphi(p)^{-1}$. By the previous theorem, $\zeta(0) = 0$ and $\zeta(p_c) = \infty$. The next theorem helps us to study the *power law behavior* of correlation length.

Theorem: $\tau_p(0, x) \leq \left(1 - \chi(p)^{-1}\right)^{|x|} \ \forall\, x$

Proof: Instead of the box $B(n)$, we consider the sphere
$$S(n) = \{x \in \mathbb{Z}^d : |x| \leq n\}$$

and its boundary

$$\partial S(n) = \{x \in \mathbb{Z}^d : |x| = n\};$$

similarly, we write $S(n,x) = x + S(n)$ and $\partial S(n,x) = x + \partial S(n)$ for the corresponding sets centered at x.

Let M_n be the number of vertices in $\partial S(n)$ which are connected to the origin by open paths, so that

$$\mathbb{E}_p(M_n) = \sum_{x \in \partial S(n)} \tau_p(0,x).$$

Similarly, we have

$$\sum_{n=0}^{\infty} \mathbb{E}_p(M_n) = \chi(p).$$

This gives us

$$\tau_p(0,z) \leq \sum_{x \in \partial S(m)} \tau_p(0,x)\tau_p(x,z)$$

whenever $|z| \geq m$. The term $\tau_p(x,z)$ is no greater than 1, so that

$$\tau_p(0,z) \leq \mathbb{E}_p(M_m) \quad \text{when } |z| \geq m.$$

We fix m and suppose that $|z| = n$. We write $n = mr + s$ for non-negative integers r, s satisfying $0 \leq s < m$. We now obtain

$$\tau_p(0,z) \leq \sum_{x \in \partial S(m)} \tau_p(0,x) \sum_{y \in \partial S(m,x)} \tau_p(x,y)\tau_p(y,z) \leq \mathbb{E}_p(M_m)^r = \mathbb{E}_p(M_m)^{\lfloor |z|/m \rfloor} \quad \text{for all } z \text{ and } m.$$

We may improve this still further as follows. Let $u \in \mathbb{Z}^d$ and let k be a positive integer. By the usual argument, we have that

$$\tau_p(0,ku) \geq \mathbb{P}_p\left(ju \leftrightarrow (j+1)u \text{ for } 0 \leq j < k\right) \leq \tau_p(0,u)^k$$

by the FKG inequality. Thus

$$\tau_p(0,u) \leq \tau_p(0,ku)^{1/k} \leq \mathbb{E}_p(M_m)^{\lfloor k|u|/m \rfloor/k} \to \mathbb{E}_p(M_m)^{|u|/m} \quad \text{as } k \to \infty$$

for all vertices u and all integers m.

Suppose now that $0 < p < p_c$, so that $\chi(p) < \infty$. There exists $m \geq 1$ such that

$$\mathbb{E}_p(M_m) \leq \left\{1 - \chi(p)^{-1}\right\}^m,$$

since, if $\mathbb{E}_p(M_m) > \left\{1 - \chi(p)^{-1}\right\}^m$ for all $m \geq 1$, then

$$\chi(p) > \sum_{m=0}^{\infty} \left\{1 - \chi(p)^{-1}\right\}^m = \chi(p),$$

a contradiction. We choose $m \geq 1$ such that

$$\mathbb{E}_p(M_m) \leq \left\{1 - \chi(p)^{-1}\right\}^m$$

and substitute this into the previous inequality to obtain

$$\tau_p(0,u) \leq \left\{1 - \chi(p)^{-1}\right\}^{|u|},$$

as claimed.

This theorem leaves us with $\varphi(p)^{-1} \leq \chi(p)$. We already know that $\chi(p)$ exhibits power law behavior near p_c. This theorem along with the observation mentioned before gives

$$\zeta(p) \sim |p - p_c|^{-\nu}$$

4.6 Mass of Percolating Cluster

The mass of percolating cluster $M(L)$ is defined to be the number of occupied sites or bonds together that span the entire system. When $p < p_c$, mass of the percolating cluster is finite and really small, we are not interested at that case. For $L \gg \xi$: The cluster appears homogeneous

$$\begin{aligned} M(L) &= (\text{No. of lattice sites}) \times (\text{Prob. site belongs to percolating cluster}) \\ &= L^d P(p) \\ &= L^d (p - p_c)^\beta \\ &= L^d \xi^{-\frac{\beta}{\nu}} \quad \text{since } \xi \propto |p - p_c|^{-\nu}. \end{aligned}$$

Now consider $L \approx \xi$, that is, we match the observations above by substituting L with ξ:

$$M(L) \propto L^D \propto \xi^D,$$

and

$$M(L) \propto L^d \xi^{-\frac{\beta}{\nu}} \propto \xi^{d-\frac{\beta}{\nu}}.$$

Therefore, we can state that

$$D = d - \frac{\beta}{\nu},$$

which is known as a *hyperscaling* relation because the Euclidean dimension d enters in the scaling relation.

Since the percolating cluster has a constant density for $L \gg \xi$, it is natural to divide the system into boxes of linear size ξ. In d dimensions, the total volume L^d will be divided into $\left(\frac{L}{\xi}\right)^d$ boxes.

Since the cluster inside each of these boxes of size ξ^d has a mass of order ξ^D, the total mass of the cluster is given by

$$M(L, \xi) = \left(\frac{L}{\xi}\right)^d \xi^D = \xi^{D-d} L^d,$$

which is, of course, equivalent to $P(p)L^d$.

In summary, we have

$$M(L, \xi) \propto \begin{cases} L^D & L \ll \xi \\ \xi^D \left(\frac{L}{\xi}\right)^d & L \gg \xi \end{cases}$$

5 Schramm Loewner Evolution

Mandelbrot considered simulations of planar simple random walk loops (simple random walks conditioned to begin and end at the same point) and looked at the "island" or "hull" formed by filling in all of the area that is disconnected from infinity. He observed that the boundary of these hulls were simple curves with fractal dimension about 4/3. Since the conjecture $\upsilon = 3/4$ suggests that paths in a scaling limit of SAWs should have dimension 4/3, he proposed the outer boundary of a Brownian loop as a possible candidate for what he called "self-avoiding Brownian motion". In particular, he conjectured that the Hausdorff dimension of the outer boundary of planar Brownian motion is 4/3. [7] proved Mandelbrot's conjecture for the dimension of the planar Brownian frontier. The proof makes use of a new process introduced by

Schramm called the *Stochastic Loewner Evolution* or *Schramm Loewner Evolution* (SLE_κ). This is a one parameter family of conformally invariant processes, indexed by κ. It turns out that conformal invariance of the Brownian hull combined with a certain "locality" property determine its distribution.

5.1 Some Initial Setup

A key conjecture in the physics of long self-avoiding walks (SAWs) is that their scaling limit is conformally invariant. Motivated by analogies to planar Brownian motion, it is natural to study such scaling limits as paths stopped upon exiting a domain, rather than focusing on time parametrization.

In particular, the conformal image of Brownian motion stopped at the exit time of a domain is again Brownian motion (modulo reparametrization), suggesting the relevance of conformal mappings to SAWs.

One expects the scaling limit of long SAWs to remain self-avoiding. Thus, we consider probability measures on continuous simple curves connecting two boundary points $z, w \in \partial D$ of a simply connected domain $D \subset \mathbb{C}$ with smooth boundary near z and w. These measures should be:

- **Conformally invariant**,
- "Uniform" over admissible curves,
- **Restriction covariant** — i.e., restriction to subdomains preserves the form of the measure.

Denote by $m^{\#}(z, w; D)$ such a probability measure on curves in D from z to w. Then:

- If $D' \subset D$ agrees with D near z and w, the conditional law of curves lying in D' is $m^{\#}(z, w; D')$.
- If $f : D \to D'$ is conformal with $f(z) = z'$, $f(w) = w'$, then the pushforward of $m^{\#}(z, w; D)$ under f equals $m^{\#}(z', w'; D')$.

Such a family is fully determined by $m^{\#}(0, \infty; \mathbb{H})$, where $\mathbb{H}$ is the upper half-plane.

Let (D, z, w) be a triplet with D a simply connected bounded domain, $z \neq w \in \partial D$, and smooth boundary near z, w. A *hull* connecting z and w is a compact subset $K \subset D \cup z, w$ such that $D \setminus K$ has exactly two components.

A **conformal restriction family** $m^{\#}(z, w; D)$ satisfies:

- *Restriction covariance*: If $D' \subset D$ agrees near z, w, then $m^{\#}(z, w; D)$ restricted to hulls in D' is proportional to $m^{\#}(z, w; D')$.
- *Conformal invariance*: If $f : D \to D'$ is conformal with $f(z) = z'$, $f(w) = w'$, then $f \circ m^{\#}(z, w; D) = m^{\#}(z', w'; D')$.

This family is characterized by $m^{\#}(0, \infty; \mathbb{H})$. For compact $A \subset \mathbb{H}$ such that $\mathbb{H} \setminus A$ is simply connected and $0 \notin A$, let Φ_A be the conformal map from $\mathbb{H} \setminus A$ to $\mathbb{H}$ fixing 0 and ∞. Then:

Lemma: There exists $a > 0$ such that for any such A,

$$\mathbb{P}[\text{hull } K \subset \mathbb{H} \setminus A] = \Phi_A'(0)^a$$

The corresponding restriction measure is denoted $\mathrm{CR}_a(z,w;D)$. Only for $a=5/8$ is the support of CR_a on simple paths.

Let $\gamma:[0,\infty)\to\mathbb{H}$ be a curve and define $H_t=\mathbb{H}\setminus\gamma[0,t]$. Then the conformal map $g_t:H_t\to\mathbb{H}$ satisfying

$$g_t(z)=z+\frac{2t}{z}+\mathcal{O}\left(\frac{1}{z^2}\right),\quad z\to\infty$$

satisfies the Loewner equation:

$$\partial_t g_t(z)=\frac{2}{g_t(z)-W_t},\quad g_0(z)=z,$$

where W_t is a continuous real-valued function. **Chordal SLE$_\kappa$** is defined by setting $W_t=\sqrt{\kappa}B_t$, where B_t is standard Brownian motion. The curve $\gamma(t)=g_t^{-1}(W_t)$ is almost surely well-defined.

Theorem: The only conformal restriction family supported on simple curves corresponds to $a=5/8$. This family is exactly chordal SLE8/3. The Hausdorff dimension of SLE8/3 is 4/3.

5.2 Scaling Limit of SAW

Let D be a connected open domain; for ease we assume D is bounded. Let z,w be distinct points in D. For each positive integer N, let z_N,w_N be lattice points closest to Nz,Nw (we may fix a convention in case of non-uniqueness).

Assume that the following stronger version of the scaling law holds: there exists a function $C(z,w;D)\in(0,\infty)$ such that

$$\lim_{N\to\infty}N^{2b}\mu_{\mathrm{SAW}}[\Lambda(z_N,w_N;D,N)]=C(z,w;D).$$

To each $\omega\in\Lambda(z_N,w_N;D,N)$, we consider $N^{-1}\omega$ as a continuous curve connecting $N^{-1}z_N$ to $N^{-1}w_N$. The specific parametrization is unimportant.

We conjecture that as $N\to\infty$, the measure $N^{2b}\mu_{\mathrm{SAW}}$, restricted to $\Lambda(z_N,w_N;D,N)$, considered as a measure on curves in D via $\omega\mapsto N^{-1}\omega$, converges weakly to a measure $m_{\mathrm{SAW}}(z,w;D)$. By construction, the total mass of this measure is $C(z,w;D)$.

Suppose ∂D is smooth, $z\in D$, and let z_N approximate z. Let $\tilde{\Lambda}(z_N;D,N)$ denote the set of SAWs $\{\omega_0,\ldots,\omega_n\}$ with $\omega_0=z_N$, $\omega_i\in ND$, and $[\omega_{n-1},\omega_n]\cap\partial(ND)\neq\emptyset$. Assume there exists a constant $C(z;D)\in(0,\infty)$ such that

$$\lim_{N\to\infty}N^{a+b}N^{-1}\mu_{\mathrm{SAW}}[\tilde{\Lambda}(z_N;D,N)]=C(z;D).$$

The factor N^{-1} accounts for the $O(N)$ boundary points at scale $1/N$.

We conjecture that as $N\to\infty$, the measure $N^{a+b}N^{-1}\mu_{\mathrm{SAW}}$, restricted to $\tilde{\Lambda}(z_N;D,N)$, considered as a measure on curves in $\overline{D}$ via $\omega\mapsto N^{-1}\omega$, converges weakly to a measure $m_{\mathrm{SAW}}(z;D)$. By construction, the total mass of this measure is $C(z;D)$.

We also assume this measure can be written as:

$$m_{\mathrm{SAW}}(z;D)=\int m_{\mathrm{SAW}}(z,w;D)\,dw,$$

where the integral is over the boundary, and $m_{SAW}(z, w; D)$ is a measure on paths connecting the interior point z to the boundary point w. We let $C(z, w; D)$ be its total measure so that

$$C(z; D) = \int_{\partial D} C(z, w; D)\, d|w|.$$

Of course, this defines $C(z, w; D)$ only for almost every $w \in \partial D$. However, we assume that $C(z, w; D)$ may be chosen to be continuous in w.

Similarly, we can consider measures connecting two distinct points $z, w \in \partial D$. Let A_1, A_2 be nontrivial, disjoint, closed subarcs of ∂D. Let $\mu_{SAW}(A_1, A_2; D, N)$ be the set of all $\omega = [\omega_0, \omega_1, \ldots, \omega_n] \in \Lambda$ such that

- $[\omega_1, \ldots, \omega_{n-1}] \subset ND$,
- $[\omega_0, \omega_1] \cap NA_1 \neq \emptyset$ and $[\omega_{n-1}, \omega_n] \cap NA_2 \neq \emptyset$.

Assume that there is a $C(A_1, A_2; D) \in (0, \infty)$ such that

$$\lim_{N \to \infty} N^{2a} N^{-2} \mu_{SAW}[\Lambda(A_1, A_2; D, N)] = C(A_1, A_2; D).$$

We conjecture that as $N \to \infty$, the restriction of the measure $N^{2a-2}\mu_{SAW}$, to $\Lambda(A_1, A_2; D, N)$, converges weakly to a measure $m_{SAW}(A_1, A_2; D)$. By construction, the total mass of this measure is $C(A_1, A_2; D)$. We also assume that this measure can be written as

$$m_{SAW}(A_1, A_2; D) = \int_{A_1} \int_{A_2} m_{SAW}(z, w; D)\, d|w|\, d|z|,$$

where $m_{SAW}(z, w; D)$ is a measure on curves connecting boundary points z and w with total mass $C(z, w; D)$ satisfying

$$C(A_1, A_2; D) = \int_{A_1} \int_{A_2} C(z, w; D)\, d|w|\, d|z|.$$

In this section, we discuss conjectures relating self-avoiding walks (SAWs) to SLE_κ. For some results, we may formulate statements such as: *if the SAW scaling limit exists and is conformally invariant, then the scaling limit is $SLE_{8/3}$.* Since the purpose of this paper is primarily to propose conjectures, we have not checked the exact hypotheses under which such conditional inferences could be justified.

Consider the conjectured scaling limit $m^*_{SAW}(z, w; D)$ as described in §3, where D ranges over simply connected domains and $z, w \in \partial D$. We summarize the conjectures about $m_{SAW}(z, w; D)$ as follows:

$$m^*_{SAW}(z, w; D) = \frac{m_{SAW}(z, w; D)}{m_{SAW}(*, *; D)}$$

This represents a probability measure associated to $m_{SAW}(z, w; D)$. The measure $m^*_{SAW}(z, w; D)$ is conjectured to belong to the CR(a) family. The only CR(a) family supported on simple curves is for $a = 5/8$. Thus, $m^*_{SAW}(z, w; D)$ is given by chordal $SLE_{8/3}$ connecting z and w in D.

Another justification is that $m^*_{SAW}(0, \infty; D)$ is conjectured to be the scaling limit of the half-plane infinite SAW. This limit should exhibit conformal invariance and the restriction property as described in §2.3. These conditions uniquely determine $SLE_{8/3}$, implying $\kappa = 8/3$.

Prediction 1. The scaling limit of the half-plane infinite SAW is chordal $SLE_{8/3}$. Moreover, the family of measures $m^*_{SAW}(z, w; D)$, where D ranges over simply connected domains and $z, w \in \partial D$, is the unique CR(5/8) family. In particular, the SAW boundary scaling exponent is:

$$a = \frac{5}{8}.$$

From §2.3, the Hausdorff dimension of $SLE_{8/3}$ paths is 4/3. Since this dimension characterizes the SAW scaling limit, we obtain:

Prediction 2. The mean-square displacement exponent for SAWs and SAPs is:

$$\nu = \frac{3}{4}.$$

Similarly, the measures $m^*_{SAP}(z, w; D)$ are conjectured to correspond to the CR(2) family. This family describes the hull of two independent Brownian excursions between boundary and interior points in D.

Prediction 3. The scaling limit of the half-plane infinite SAP is the outer boundary of a Brownian excursion from 0 to ∞ in $\mathbb{H}$. The family of measures $m^*_{SAP}(z, w; D)$, where D ranges over simply connected domains and $z \in \partial D, w \in D$, is the unique CR(2) family. The *chordal* SLE_κ is given by

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa} B(t)}$$

The parameter $B(t)$ stands for standard one dimensional brownian motion.

6 Verification through Simulations

Here, we have simulated SAW and its physical properties that has already been described analytically. The algorithms used have also been mentioned alongside the corresponding plots and visuals. All the simulations has been performed in R 4.4.0 in a system with Windows 11 Operating System, AMD Ryzen 5 4600H @ 3.0 GHz, 16 GB DDR4, 4GB NVIDIA GTX 1650.

Algorithm 1 Pivot Algorithm for Self-Avoiding Walks (SAWs)

Input:
  L - Length of the walk
  steps - Number of pivot steps to perform
Output:
  Final SAW configuration as $L + 1 \times 2$ matrix
function SimulateSAWPivot(L, *steps*)
    Define directions: $\boldsymbol{d} = \{(1,0), (-1,0), (0,1), (0,-1)\}$
    // Initialize straight horizontal walk
    $\boldsymbol{walk} \leftarrow \text{zeros}(L+1, 2)$
    for $i \leftarrow 2$ to $L + 1$ **do**
        $\boldsymbol{walk}[i] \leftarrow \boldsymbol{walk}[i-1] + (1,0)$
    end for
    Define rotation matrices:
    $\boldsymbol{R} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$
    function HasIntersection($\boldsymbol{path}$)
        $\text{pos\_strs} \leftarrow \{\text{paste}(\boldsymbol{p}) \text{ for } \boldsymbol{p} \text{ in } \boldsymbol{path}\}$
        return $|\text{pos\_strs}| \neq |\text{unique}(\text{pos\_strs})|$
    end function
    // Perform pivot steps
    for $i \leftarrow 1$ to *steps* **do**
        $\mathit{pivot} \leftarrow \text{RandomInteger}(2, L)$ ▷ Exclude endpoints
        $\boldsymbol{rot} \leftarrow \text{RandomChoice}(\boldsymbol{R})$
        $\boldsymbol{fixed} \leftarrow \boldsymbol{walk}[1 : \mathit{pivot}]$
        $\boldsymbol{moving} \leftarrow \boldsymbol{walk}[(\mathit{pivot}+1) : (L+1)]$
        $\boldsymbol{pivot\_point} \leftarrow \boldsymbol{walk}[\mathit{pivot}+1]$
        $\boldsymbol{translated} \leftarrow \boldsymbol{moving} - \boldsymbol{pivot\_point}$
        $\boldsymbol{rotated} \leftarrow \boldsymbol{translated} \times \boldsymbol{rot}^\top$
        $\boldsymbol{new\_part} \leftarrow \boldsymbol{rotated} + \boldsymbol{pivot\_point}$
        $\boldsymbol{proposed} \leftarrow \text{concatenate}(\boldsymbol{fixed}, \boldsymbol{new\_part})$
        if $\neg$HasIntersection($\boldsymbol{proposed}$) **then**
            $\boldsymbol{walk} \leftarrow \boldsymbol{proposed}$
        end if
    end for
    return $\boldsymbol{walk}$
end function

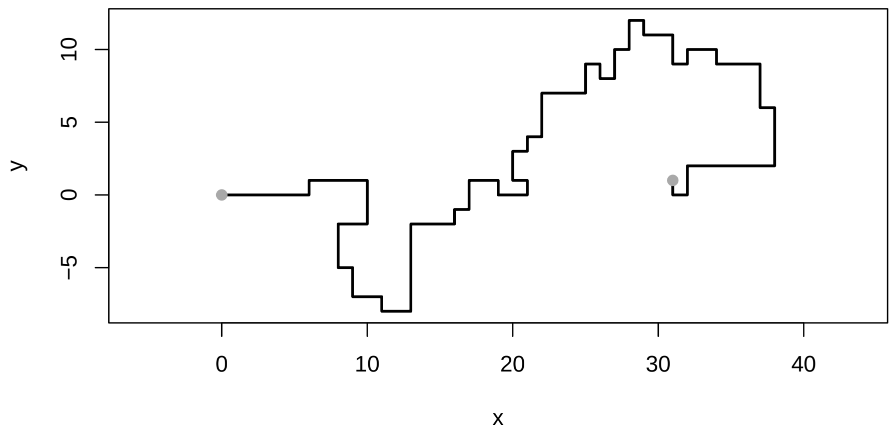


Figure 1: SAW Using Pivot Algorithm

Algorithm 2 Count Self-Avoiding Walks (SAWs)

```
Input:
  pos - Current position (2D coordinates)
  visited - Set of visited positions
  steps_remaining - Number of steps remaining
Output:
  Count of SAWs starting from pos with steps_remaining steps
function CountSAWs(pos, visited, steps_remaining)
    if steps_remaining = 0 then
        return 1                                   ▷ Base case: found a valid SAW
    end if
    count ← 0
    Define possible moves: {↑= (0, 1), ↓= (0, −1), ←= (−1, 0), →= (1, 0)}
    for each move in possible moves do
        new_pos ← pos + move
        key ← Concatenate(new_pos[0], new_pos[1])  ▷ Create unique key for position
        if key ∉ visited then
            new_visited ← visited ∪ {key}
            count ← count + CountSAWs(new_pos, new_visited, steps_remaining − 1)
        end if
    end for
    return count
end function
```

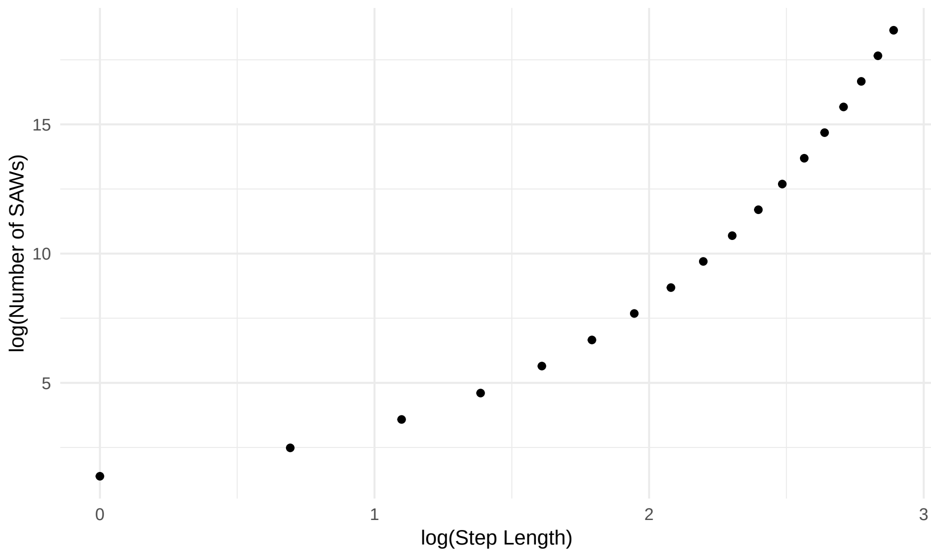
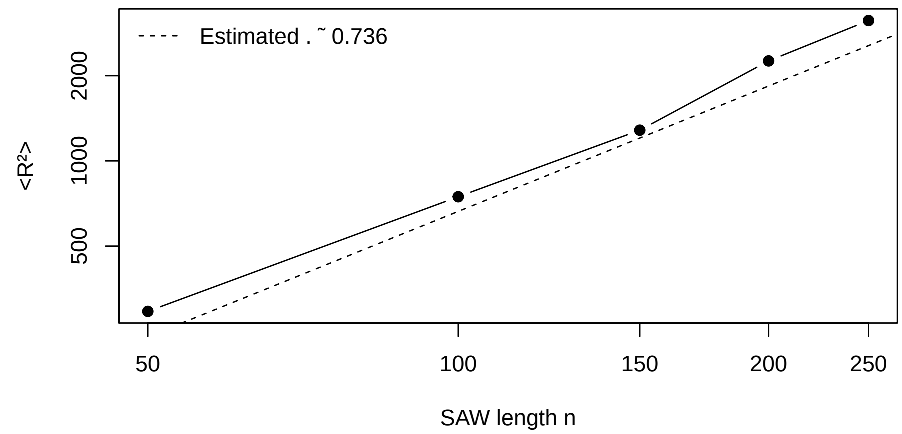


Figure 2: Number of SAW of Particular Length (Increasing exponentially)

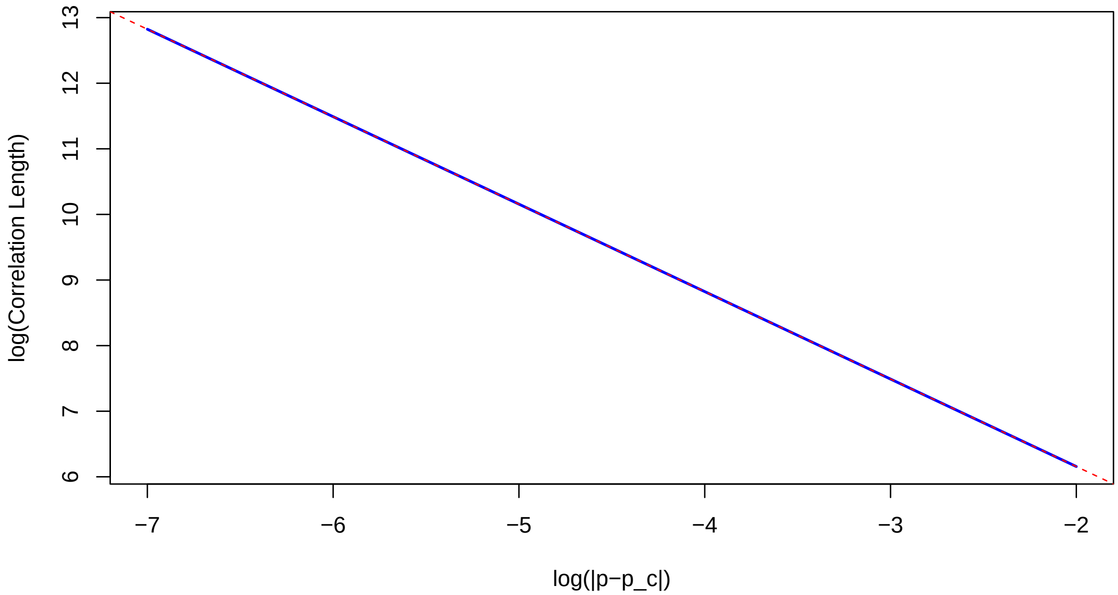
Algorithm 3 Mean Squared Displacement for SAW

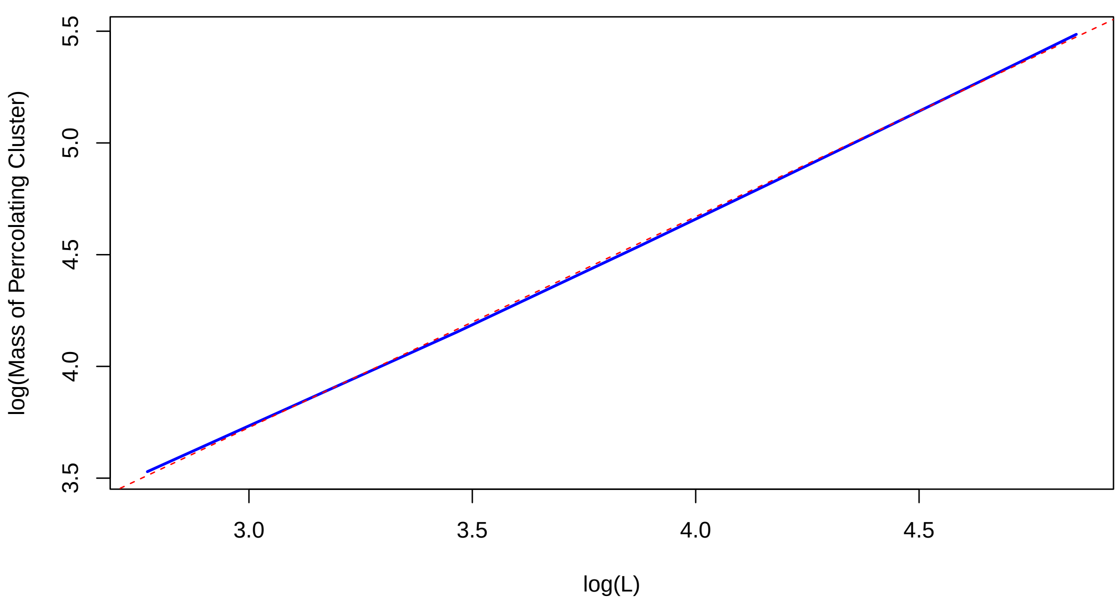
function EndToEndSquared($\boldsymbol{walk}$)
    $\Delta x \leftarrow \boldsymbol{walk}[n][1] - \boldsymbol{walk}[1][1]$
    $\Delta y \leftarrow \boldsymbol{walk}[n][2] - \boldsymbol{walk}[1][2]$
    return $\Delta x^2 + \Delta y^2$
end function
Main Procedure:
$lengths \leftarrow$ [list of walk lengths to study]
$trials \leftarrow$ number of trials per length
$\boldsymbol{R2\_means} \leftarrow$ zero vector of size $|lengths|$
for $i \leftarrow 1$ to $|lengths|$ **do**
    $n \leftarrow lengths[i]$
    $\boldsymbol{R2} \leftarrow$ zero vector of size $trials$
    for $j \leftarrow 1$ to $trials$ **do**
        $\boldsymbol{walk} \leftarrow$ SimulateSAWPivot(i, n)
        $\boldsymbol{R2}[j] \leftarrow$ EndToEndSquared($\boldsymbol{walk}$)
    end for
    $\boldsymbol{R2\_means}[i] \leftarrow$ mean($\boldsymbol{R2}$)
end for



Algorithm 4 Estimate Correlation Length Exponent ν in 2D Percolation

Input: Lattice size L, number of trials T, percolation threshold $p_c = 0.5$.
Output: Estimated exponent ν
function EstimateCorrelationLength(L, p, T)
    Initialize empty array `distances`
    for $t = 1$ to T **do**
        Create empty graph G with $L \times L$ nodes
        for $i = 1$ to L **do**
            for $j = 1$ to L **do**
                if $i < L$ and random number $< p$ **then**
                    Add edge between (i, j) and $(i + 1, j)$
                end if
                if $j < L$ and random number $< p$ **then**
                    Add edge between (i, j) and $(i, j + 1)$
                end if
            end for
        end for
        Identify connected components in G
        Select only finite nontrivial clusters: $1 <$ size $< 0.2L^2$
        if no such cluster exists **then**
            continue to next trial
        end if
        Randomly select one such cluster
        Compute center of mass (c_x, c_y) of cluster
        Compute radius $r = \sqrt{\frac{1}{n}\sum_{i=1}^{n}((x_i - c_x)^2 + (y_i - c_y)^2)}$
        Store r in `distances`
    end for
    return mean of nonzero values in `distances`
end function


log(Correlation Length)
13
12
11
10
9
8
7
6
−7
−6
−5
−4
−3
−2
log(|p−p_c|)


log(Mass of Perrcolating Cluster)
5.5
5.0
4.5
4.0
3.5
3.0
3.5
4.0
4.5
log(L)

Algorithm 5 Estimate Fractal Dimension of Percolating Cluster via Log-Log Scaling

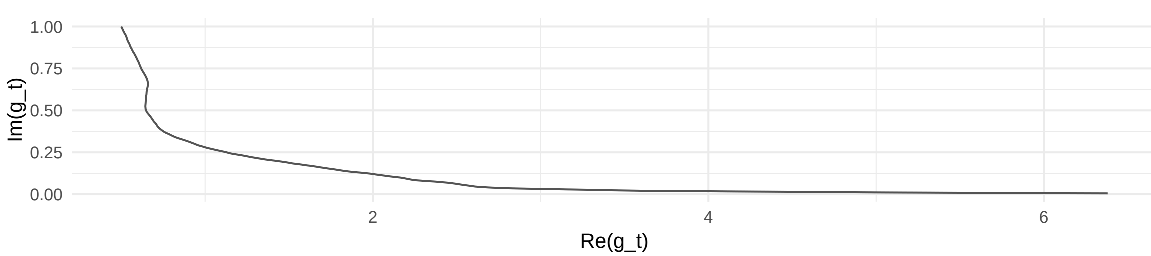
```
Input: Percolation threshold p_c = 0.5.
Output: Estimated slope (fractal dimension) D
function INDEX(x, y, L)
    return (x − 1) · L + y
end function
function GET_BONDS(L)
    Initialize empty list bonds
    for i = 1 to L do
        for j = 1 to L do
            id ← INDEX(i, j, L)
            right ← INDEX(i, j + 1 mod L, L)
            down ← INDEX(i + 1 mod L, j, L)
            Add edge (id, right) and (id, down) to bonds
        end for
    end for
    return bonds
end function
function IS_SPANNING(component_nodes, L)
    Compute x and y coordinates from node indices
    return TRUE if component touches all four boundaries
end function
function SIMULATE_MASS(L, p, n_trials)
    masses ← empty array
    bonds ← GET_BONDS(L)
    for t = 1 to n_trials do
        Randomly activate bonds with probability p
        Construct graph from active bonds
        Identify connected components
        Initialize max_mass ← 0
        for each component do
            if IS_SPANNING(component, L) then
                mass ← size of component
                if mass > max_mass then
                    max_mass ← mass
                end if
            end if
        end for
        Append max_mass to masses
    end for
    return mean of nonzero values in masses
end function
```

Algorithm 6 Simulate Loewner Evolution $g_t(z_0)$ with SLE$_\kappa$ using RK4

Input: SLE parameter $\kappa = \frac{8}{3}$, total time $T = 5$, steps $N = 1000$, initial point $z_0 = 0.5 + i$
Output: Trajectory of $g_t(z_0)$ in the complex plane
Set time step $\Delta t \leftarrow T/N$
Define time grid $t_0, t_1, \ldots, t_N$
Generate Brownian motion:
    For each $i = 1$ to N: $dB_i \leftarrow \sqrt{\Delta t} \cdot \mathcal{N}(0, 1)$
    Compute $B_t \leftarrow \sum dB_i$ (cumulative sum)
    Set driving function $\xi_t \leftarrow \sqrt{\kappa} \cdot B_t$
Initialize $g_0(z_0) \leftarrow z_0$
Define ODE: $f(g_t, \xi_t) = \frac{2}{g_t - \xi_t}$
for $i = 1$ to $N - 1$ **do**
    $\xi_t \leftarrow \xi[i]$, $\xi_{t+\Delta t} \leftarrow \xi[i + 1]$
    Compute Runge-Kutta terms:
    $k_1 \leftarrow \Delta t \cdot f(g[i], \xi_t)$
    $k_2 \leftarrow \Delta t \cdot f(g[i] + \frac{k_1}{2}, \xi_{t+\Delta t})$
    $k_3 \leftarrow \Delta t \cdot f(g[i] + \frac{k_2}{2}, \xi_{t+\Delta t})$
    $k_4 \leftarrow \Delta t \cdot f(g[i] + k_3, \xi_{t+\Delta t})$
    Update:

$$g[i + 1] \leftarrow g[i] + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

end for



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